# On Practical Fixed-Time Convergence for Differential Riccati Equations

Sergiy Zhuk, Andrey Polyakov

# On Practical Fixed-Time Convergence for Differential Riccati Equations*

Sergiy Zhuk and Andrey Polyakov†

**Abstract.** Sufficient conditions for fixed-time convergence of matrix differential Riccati equations towards an ellipsoid in the space of symmetric non-negative matrices are proposed. These conditions are based on the classical concept of uniform complete observability. The fixed-time convergence is demonstrated for the Riccati matrix and its inverse. This convergence is then used to design a globally convergent observer for bilinear chaotic differential equations (e.g. equations with zero Lyapunov exponents). Convergence of the observer is confirmed by numerical experiments with ODEs obtained by finite-difference discretization of a hyperbolic PDE in 1D (Burgers-Hopf equation).

**Key words.** nonlinear filtering; uniform observability; Lyapunov exponents; bilinear systems; Riccati equations; fixed-time convergence

**AMS subject classifications.** 62M20; 37C50; 34D06; 37M25

## 1. Introduction.

Nonlinear filtering and observer design are fundamental in diverse fields including synchronization in complex networks, data assimilation and control engineering to name just a few. Theoretically, solution of the stochastic filtering problem for Markov diffusions is given by the so-called Kushner-Stratonovich (KS) equation [7], a stochastic Partial Differential Equation (PDE) which describes evolution of the conditional density of the states of the underlying diffusion process. For linear systems, KS equation is equivalent to the Kalman-Bucy Filter (KF) [6].

In contrast, deterministic state estimators, including the algorithm presented in this paper, assume that errors have bounded energy and belong to a given bounding set. The state estimate is then defined as the minimax center of the reachability set, and temporal dynamics of the minimax center is described by a minimax filter [8]. The latter may be constructed by using dynamic programming, i.e., the set $V \leq 1$, where $V$ is the so-called value function $V$ solving a Hamilton-Jacobi-Bellman (HJB) equation [3], coincides with the reachability set. For linear dynamics, equations of the minimax filter coincide with those of KF [Krener(1980)]. In particular, dynamics of the state error covariance matrix of the KF is given by the solution of matrix differential Riccati equation, and its inverse defines the shape of the reachability set (ellipsoid) of the linear ordinary differential equations (ODEs).

For generic nonlinear models both minimax and stochastic filters are infinite-dimensional, i.e., to get an optimal estimate one needs to solve a PDE (KF or HJB). Hence, if the state space is of high dimension then both filters become computationally intractable due to the

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"curse of dimensionality". To compute the filter one usually compromises optimality to gain computationally tractable approximations (e.g. ensemble KF [4]). In fact, there is a deep relationship between observers and (optimal) minimax filters: in the linear case the minimax/Kalman filter uniformly converge to the observer if the observational noise/model error "disappears" as $t \to \infty$; see [2, Frank and Zhuk(2018)]. In what follows we build on this relationship and design an observer, approximating the optimal minimax filter (and implicitly a solution of HJB), by revealing connections between fixed-time convergence of matrix differential Riccati equations, and exponential convergence of a certain class of observers for bilinear equations. Conditions for fixed-time convergence together with observer design for bilinear systems represent our key contribution.

More specifically, an important feature of the Riccati equation studied in this paper is its practical fixed-time convergence (see e.g. [Polyakov(2012)]), which means that independently of the initial condition the state of the differential Riccati equation converges to a neighborhood of zero in a fixed time. We also show that the uniform complete observability [Bucy(1972)] along the observer trajectory (unlike the standard assumption of uniform complete observability of the "true trajectory", e.g. [Tranninger et al.(2019)Tranninger, Seeber, Zhuk, Steinberger, and Horn]) is the sufficient condition for fixed time convergence for the Riccati equation. The latter means that the state of the system can always be estimated in a fixed time using the observer proposed below. Moreover, in the disturbance-free case we prove exponential convergence of the observer to the state of the bilinear system.

Bilinear ODEs, considered below, represent a large class of practically important models in fluid dynamics (Navier-Stokes and Euler equations [10, 5]), transportation domain (LWR model) [9], chaotic dynamical systems [11] and even weather and ocean modelling [12]. For example, the bilinear ODE studied in this paper can be obtained as a Galerkin projection of the Euler equation [5], or a finite-difference discretization of the Burgers equation (as discussed below). State estimation for bilinear systems is a challenging problem especially in the case of high dimension of the state vector. Indeed, in the latter case it is hard to implement the differential algebraic approach conventional for nonlinear systems [Fliess et al.(1995)Fliess, Lévine, Martin, and Rouchon, Isidori(1995), Hermann and Krener(1977)]. Our design does not rely upon linearization, used in the popular framework of Extended KF. Instead we make use of the fixed-time convergence to prove that the Riccati matrix defines the Lyapunov function for the estimation error equation, and then we study its robustness w.r.t. Lipschitz perturbations [1]. As a result, equations of our observer are different to that of Extended KF. We note that the latter may diverge for nonlinear equations with non-negative Lyapunov exponents which are studied below, and specifically for Burgers equations which have only zero Lyapunov exponents [Frank and Zhuk(2018)].

This paper is organised as follows: Section 2 contains all the notations and preliminary information, Section 3 contains main results on fixed time convergence (Theorem 3.1) and observer design (Theorem 3.8). Numerical experiments are given in Section 4.

## 2. Mathematical preliminaries.

*Notation.* $\mathbb{R}^n$ – Euclidean space of $n$-dimensional column vectors with real-valued entries and canonical basis $\{\psi_1 \dots \psi_n\}$; $C(t_0, T, \mathbb{R}^n)$ – the space of continous $\mathbb{R}^n$-valued functions; $\mathbb{S}^n$ – Hilbert space of symmetric non-negative definite $n \times n$-matrices. $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote minimal and maximal eigen-values of a matrix $P$. $\mathrm{Tr}(P)$ – trace of $P$; $I_n \in \mathbb{S}^n$ – the $n \times n$-identity matrix;

*Trace Inequalities.* If $A$ is symmetric and $B$ is skew-symmetric then $\mathrm{Tr}(AB + B^\top A) = 0$.

Lemma 2.1. *If* $0 < A \in \mathbb{S}^n$ *then*

$$\|A\|_\infty = \max_{ij} |a_{ij}| \le \mathrm{Tr}\, A \le \sqrt{n \,\mathrm{Tr}\, A^2}.$$

*Proof.* I. Recall that $\langle A, B\rangle = \mathrm{Tr}(AB)$ defines an inner product on $\mathbb{S}^n$, and by Schwarz inequality we find that $|\langle A, B\rangle| \le \langle A, A\rangle^{\frac{1}{2}} \langle B, B\rangle^{\frac{1}{2}}$. Hence, for $B = I_n$ and $A > 0$ we have $\mathrm{Tr}(A) \le \sqrt{n\,\mathrm{Tr}(A^2)}$.

II. Let $n = 2$. If $A \in \mathbb{S}^n$ is positive definite then from Silvester Criterion we derive

$$a_{11} > 0, a_{22} > 0, a_{11}, a_{22} > a_{12}^2.$$

In this case $a_{ii} \le \mathrm{Tr}\, A$ and

$$\max_{ij} |a_{ij}| \le \max\{a_{11}, \sqrt{a_{11}a_{22}}, a_{22}\} \le \mathrm{Tr}\, A.$$

Let $n > 2$ and $A > 0$ be represented

$$A = \begin{pmatrix} A_{n-1} & \tilde{a} \\ \tilde{a}^\top & a_{nn} \end{pmatrix},$$

where $A_{n-1} \in \mathbb{S}^{n-1}$, $\tilde{a} \in \mathbb{R}^{n-1}$. Let us assume, by induction, that $\|A_{n-1}\|_\infty \le \mathrm{Tr}\, A_{n-1}$. Using Shur Complement we conclude

$$a_{nn}A_{n-1} - \tilde{a}\tilde{a}^\top > 0.$$

Hence, for the basis vector $\psi_i = (0, ..., 0, 1, 0, ....0)^\top$ in $\mathbb{R}^{n-1}$ we have

$$0 < \psi_i^\top \left(a_{nn}A_{n-1} - \tilde{a}\tilde{a}^\top\right) \psi_i = a_{nn}a_{ii} - a_{in}^2.$$

Hence, $a_{in}^2 \le \mathrm{Tr}^2 A, i = 1, ..., n-1$ and $\|A\|_\infty \le \mathrm{Tr}\, A$. ∎

*Uniform complete observability .* Consider an LTV system

(1) $$\dot{x}(t) = A(t)x(t)\,, \quad x(t_0) = x_0 \in \mathbb{R}^n$$

and assume that the output $y$ of this system is given in the following form:

(2) $$y(t) = C(t)x(t)\,, \quad t \in [t_0, T].$$

We assume that matrix-valued functions $A$ and $C$ are bounded on $[t_0, +\infty)$. Let $X$ denote the unique matrix-valued solution of the following differential equation:

$$\dot{X}(t) = A(t)X(t)\,, \quad X(t_0) = X_0 \in \mathbb{R}^{n\times n}\,. \tag{3}$$

Define $\Phi(t,s) := X(t)X^{-1}(s)$. If $x$ solves (1) then it verifies the following identity:

$$x(s) = \Phi(s,t)x(t)\,, \quad \forall t, s \geq t_0$$

and, in particular, we can write:

$$y(s) = C(s)\Phi(s,t)x(t)\,, \quad \forall s \in [t_0, t], \quad t \leq T.$$

Now, if we multiply the above equation by $(C(s)\Phi(s,t))^\top$ and integrate w.r.t. parameter $s$ from $t_0$ to $t$ we get:

$$\mathcal{N}(t,t_0)x(t) = \tilde{y}(t)\,, \quad t \in (t_0, T]. \tag{4}$$

provided $\tilde{y}(t) := \int_{t_0}^t \Phi^\top(s,t)C^\top(s)y(s)ds$ and

$$\mathcal{N}(t,t_0) := \int_{t_0}^t \Phi^\top(s,t)C^\top(s)C(s)\Phi(s,t)ds\,.$$

Clearly, if $\mathcal{N}(t,t_0)$ is of full rank then $x(t) = \mathcal{N}^{-1}(t,t_0)\tilde{y}(t)$, i.e. we can obtain exact state vector $x(t)$ from the output (2).

Definition 2.2 ([Bucy(1972)]). *LTV system* (1) *is called uniformly completely observable if there exist* $\nu > 0$ *and* $\alpha, \beta > 0$ *such that*

$$\alpha I < \mathcal{N}(t, t-\sigma) < \beta I, \quad \forall \tau \geq t_0\,.$$

It is easy to show that $\mathcal{N}(t,t_0)$ verifies the following differential Lyapunov equation: $\mathcal{N}(t_0,t_0) = 0$ and

$$\dot{\mathcal{N}}(t,t_0) = -A^\top\mathcal{N}(t,t_0) - \mathcal{N}(t,t_0)A + C^\top(t)C(t),. \tag{5}$$

*Fixed-time stability.* Recall [Polyakov(2012)] that $\Omega \subset \mathbb{R}^n$ is said to be *globally uniformly fixed-time attractive* set of the system

$$\dot{\xi}(t) = f(t,\xi(t)), \quad t > t_0 \tag{6}$$

if there exists $T_{\max} > 0$ such that

$$\xi(t) \in \Omega, \quad \forall t \geq t_0 + T_{\max},$$

for any initial value $\xi(t_0) \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}$.

Below, in particular, we show that the standard Riccati equation of many $H_2$-type filters (Kalman filter, minimax filter, least-squares filter etc) has this property.

*Exponential convergence and bounded perturbations.* Following [1, Theorem 4.4.6.] we introduce

**Lemma 2.3.** *Let*

1) *system* (1) *be uniformly asymptotically stable, i.e.*

$$\|\Phi(t,s)\| \le Ke^{-\alpha(t-s)}\,, \quad t \ge s \ge 0, \quad K>0, \alpha>0,$$

2) $\|\Delta(t)\| \le \delta$ *for* $t \ge 0$ *and* $\delta > 0$

*Then the fundamental matrix* $\tilde{\Phi}$ *of the perturbed system* $\dot{x} = A(t)x + \Delta(t)x$ *verifies the following inequality:*

$$\|\tilde{\Phi}(t,s)\| \le Ke^{-\beta(t-s)}\,, \quad t \ge s \ge 0$$

*where* $\beta = \alpha - \delta K$. *If* $\beta > 0$ *then the perturbed system is uniformly asymptotically stable.*

This result shows how to compute the Lyapunov spectrum of a perturbed LTV system $\dot{x} = A(t)x + \Delta(t)x$ provided that the original LTV $\dot{x} = A(t)x$ is uniformly exponentially stable, and the perturbation $\Delta$ is bounded.

## 3. Main results.

Let $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$ denote the state vector and output of the following bi-linear system:

$$\dot{x}(t) = B(x(t))\,x(t), \quad x(t_0) = x_0\,, \tag{7}$$

$$y(t) = C(t)x(t) \tag{8}$$

provided $B(x) = -B^\top(x)$ and $x \mapsto B(x)$ is a linear mapping, and $C(t) \in \mathbb{R}^{m\times n}$ is a given measurable matrix-valued function such that

$$\lambda_{\max}(C^\top(t)C(t)) \le \bar{c} < +\infty\,.$$

Consider the following system of equations:

$$\dot{z} = B(z)z + PC^\top R(y - Cz), \qquad z(t_0) = z_0 \tag{9}$$

$$\dot{P} = B(z)P + PB^\top(z) - PC^\top RCP + Q, \qquad P(t_0) = P_0 \tag{10}$$

where $R = R^\top$ and $Q = Q^\top$ are given continuous matrix-valued functions such that

$$0 < \underline{r} \le \lambda_{\min}(R(t)) \le \lambda_{\max}(R(t)) \le \bar{r} < +\infty, \tag{11}$$

$$0 < \underline{q} \le \lambda_{\min}(Q(t)) \le \lambda_{\max}(Q(t)) \le \bar{q} < +\infty\,. \tag{12}$$

In the forthcoming sections we report the following results:

- existence and uniqueness of the unique continuous solution for the system (9)-(10) on $[t_0, +\infty)$ (Theorem 3.1)
- practical fixed-time stability of the Riccati equation (Corollary 3.4)
- observer design and uniform exponential stability for the estimation error (Theorem 3.8)

### 3.1. Existence, uniquness and practical fixed-time convergence for Riccati equation.

Theorem 3.1. *Let $P(t_0) > 0$ and $z(t_0) \in \mathbb{R}^n$. Then system (9)-(10) has the unique solution $(z, P)$ such that $z \in C(t_0, T, \mathbb{R}^n)$ and $P \in C(t_0, T, \mathbb{S}^n)$ for any $T < +\infty$. Moreover, $P(t) > p_1 I$ for some $p_1 > 0$.*

*Proof.* Since $B$ is skew-symmetric then $\frac{d}{dt}x^\top(t)x(t) = 0$, i.e. $\|x(t)\| = \|x(t_0)\| = C < +\infty$ for all $t \geq t_0$. Taking into account that the right-side of the system (9)-(10) is locally Lipschitz continuous w.r.t. state variables and time variable we conclude (e.g. by using standard argument based on Picard theorem) that for any $P(t_0) = P^\top(t_0) > 0$ this system has the unique solution $(z, P)$ defined on $[t_0, T)$ such that $0 < P(t) = P^\top(t)$ for all $t \in [t_0, T)$, where $T > t_0$ is given by one of the following cases:

1) $T = +\infty$;
2) $T < +\infty$, $\sup_{t\in[t_0,T)} \|z(t)\| < +\infty$ and $\|P(t)\| \to +\infty$ as $t \to T$;
3) $T < +\infty$, $\|z(t)\| \to +\infty$ as $t \to T$ and $\sup\limits_{t\in[t_0,T)} \|P(t)\| < +\infty$;
4) $T < +\infty$, $\|z(t)\| \to +\infty$ and $\|P(t)\| \to +\infty$ as $t \to T$;
5) $T < +\infty$, $\sup_{t\in[t_0,T)} \|z(t)\| < +\infty$, $\sup_{t\in[t_0,T)} \|P(t)\| < +\infty$ and $P(T)$ is not positive definite.

Let us now prove that case 1) is the only possible one, i.e. $T = +\infty$ for any $z_0 \in \mathbb{R}^n$ and $P_0 > 0$. Suppose the contrary.

**Case 2).** In this case, $\|B(z)\| \leq \|B\|\|z\|_\infty < +\infty$ and we can write (recall that $\mathrm{Tr}(B(z)P + PB^\top(z)) = 0$):

$$\frac{d}{dt}\mathrm{Tr}(P) = -\mathrm{Tr}(PC^\top RCP) + \mathrm{Tr}(Q)$$

and

$$\mathrm{Tr}\, P(t) \leq \mathrm{Tr}\, P_0 + n\bar{q}(T - t_0) < +\infty, \quad \forall t \in [t_0, T].$$

Hence, using Lemma 2.1 we derive the contradiction.

**Case 3).** If

$$V_0 = z^\top P^{-1} z$$

then

$$\begin{aligned}
\dot{V}_0 =& - z^\top C^\top RCz + 2z^\top C^\top RCx - z^\top P^{-1}QP^{-1}z^\top \\
=& - z^\top C^\top RCz + 2z^\top C^\top RCx - (x)^\top C^\top RCx + \\
& x^\top C^\top RCx - z^\top P^{-1}QP^{-1}z^\top \\
=& - (z - x)^\top C^\top RC(z-) + x^\top C^\top RCx - z^\top P^{-1}QP^{-1}z \\
\leq& x^\top C^\top RCx - z^\top P^{-1}QP^{-1}z^\top \\
\leq& x^\top C^\top RCx - \lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})V_0(z)
\end{aligned} \tag{13}$$

By assumption, $\|P(t)\|$ is uniformly bounded on $[t_0, T)$. Hence $\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}) > c > 0$ for a positive number $c$. (13) implies (e.g. using Bellman-Gronwall lemma) that $V_0(z(T)) < +\infty$, and hence $\|z(T)\| < +\infty$.

**Case 4)**. Let $V$ be defined as follows

$$V(z, P) = z^\top P^{-1} z + \mathrm{Tr}(P).$$

We have:

$$\begin{aligned}
\dot{V} =& - z^\top C^\top RCz + 2z^\top C^\top RCx - z^\top P^{-1}QP^{-1}z^\top + \\
& \mathrm{Tr}(Q) - \mathrm{Tr}(PC^\top RCP) \\
=& - (z-x)^\top C^\top RC(z-x) + x^\top C^\top RCx + \\
& \mathrm{Tr}(Q) - z^\top P^{-1}QP^{-1}z^\top - \mathrm{Tr}(PC^\top RCP) \\
\leq& x^\top C^\top RCx + \mathrm{Tr}(Q)
\end{aligned} \tag{14}$$

and $V(t) \leq V(t_0) + \mathrm{Tr}(Q)T + \int_{t_0}^T x^\top C^\top RCx dt < +\infty$ for all $t \in [0, T]$. Using Lemma 2.1 we derive the contradiction.

**Case 5)**. By assumption, $P(t) > 0$ on $[t_0, T)$ and $P(T)x = 0$ for some $x \neq 0$. Hence, $W = P^{-1}(t)$ is defined for every $t \in [t_0, T)$, and $\lim_{t\uparrow T} \|W(t)\| = +\infty$. In this case, $\mathrm{Tr}\, W(t) \to \infty$ as $t \to T$ due to Lemma 2.1. For any $t \in [t_0, T)$ we have:

$$\dot{W} = -W\dot{P}W = -B^\top(z)W - WB(z) - WQW + C^\top RC,$$

$W(t_0) = P^{-1}(t_0)$. Since $\mathrm{Tr}(A) = \sum_{j=1}^n \psi_i^\top A\psi_i$ and

$$\mathrm{Tr}(WQW) = \sum_{j=1}^n \psi_i^\top (WQW)\psi_i \geq \lambda_{\min}(Q) \sum_{j=1}^n \psi_i^\top W^2 \psi_i$$

it follows from Lemma 2.1 that

$$\begin{aligned}
\frac{d}{dt} \mathrm{Tr}\, W &= \mathrm{Tr}(C^\top RC) - \mathrm{Tr}(WQW) \\
&\leq \mathrm{Tr}(C^\top RC) - \frac{\lambda_{\min}(Q)}{n} \mathrm{Tr}^2(W) \\
&\leq \bar{r}\bar{c}n - \tfrac{q}{n} \mathrm{Tr}^2(W).
\end{aligned}$$

Hence, $\dfrac{d}{dt} \mathrm{Tr}\, W < 0$ if $\mathrm{Tr}(W) > \sqrt{\bar{r}\bar{c}n^2/\underline{q}}$, i.e. $\mathrm{Tr}(W(t))$ cannot tend to $\infty$ as $t \to T$. We derive the contradiction. Therefore, the obtained inequalities hold for all $t > 0$ and imply

$$\mathrm{Tr}\, P^{-1}(t) \leq \max\left\{\mathrm{Tr}\, P^{-1}(0), \sqrt{\bar{r}\bar{c}n^2/\underline{q}}\right\}$$

and

$$\lambda_{\min}(P(t)) \geq \frac{1}{\max\left\{\mathrm{Tr}\, P^{-1}(0), \sqrt{\bar{r}\bar{c}n^2/\underline{q}}\right\}} > 0.$$

■

**3.2. Fixed-time convergence of Riccati equation.** Now, let us demonstrate practical fixed-time convergence of the Riccati equation. To this end we introduce

Definition 3.2. *Let $z$ solve (9), and let $\dot X = B(z)X$, $X(0) = X_0$, and set $\Phi(t, s) = X(t)X^{-1}(s)$. $\mathcal{N}(t, t-\sigma; R, z)$ is said to be* a gramian along a trajectory $z$ of (9) *if $\mathcal{N}(t, t-\sigma; R, z) = \int_{t-\sigma}^t \Phi^\top(s, t)C^\top RC\Phi(s, t)ds$.*

Corollary 3.3. *Let $z$ and $P$ solve (9)-(10) on $[t_0, +\infty)$, and let $\mathcal{N}(t, t-\sigma; R, z)$ denote the gramian along the trajectory $z$. If there exist $\alpha > 0$ and $\sigma > 0$ such that for any $t > t_0 + \sigma$ we have that $\alpha I < \mathcal{N}(t, t-\sigma; I, z)$ then*

$$\lambda_{\max}(P(t)) \le \frac{1}{\underline{r}\alpha} + \int_{t-\sigma}^t \lambda_{\max}(Q)ds\,, \quad t \ge t_0 + \sigma. \tag{15}$$

*Proof.* Assume that $t < +\infty$ and consider the standard LQR design problem for the system $\dot q = -B^\top(z)q + C^\top u$, $q(t) = h$ with the cost $J(u) = q^\top(t_0)P(t_0)q(t_0) + \int_{t_0}^t u^\top R^{-1}u + q^\top Qqds$. It is known that the feed-back $\hat u = RCPq$ minimizes $J$ and $J(\hat u) = h^\top P(t)h$. Hence for any other control $u$ we have that $J(\hat u) = h^\top P(t)h \le J(u)$. Let us select $u$ as follows: set $u(s) = RC(s)\Phi(s, t)\mathcal{N}^{-1}(t, t-\sigma; R, z)h$ for $s \in [t-\sigma, t]$ and set $u(s) = 0$ for $s \in [t_0, t-\sigma)$. Let us show that for this $u$ one gets: $q(s) = 0$ for all $s \in [t_0, t-\sigma]$. To this end recall that

$$q(s) = \Phi^\top(t, s)h - \int_s^t \Phi^\top(\tau, s)C^\top(\tau)u(\tau)d\tau$$

and so $q(t-\sigma) = 0$ if[1]

$$\Phi^\top(t, t-\sigma)h = \Phi^\top(t, t-\sigma)\int_{t-\sigma}^t \Phi^\top(\tau, t)C^\top(\tau)u(\tau)d\tau$$

Clearly, for the above choice of $u$ we get that $q(s) = 0$ for all $s \in [t_0, t-\sigma]$. Hence, for this $u$ we get that

$$\begin{aligned} J(\hat u) = h^\top P(t)h &\le \int_{t-\sigma}^t u^\top R^{-1}u + q^\top Q^{-1}qds \\ &= h^\top\mathcal{N}^{-1}(t, t-\sigma; R, z)h + \int_{t-\sigma}^t q^\top Qqds \\ &\le \frac{\|h\|^2}{\underline{r}\alpha} + \int_{t-\sigma}^t q^\top Qqds \end{aligned} \tag{16}$$

since $\mathcal{N}(t, t-\sigma; R, z) \ge \underline{r}\mathcal{N}(t, t-\sigma; I, z) \ge \underline{r}\alpha$. To compute $\int_{t-\sigma}^t q^\top Qqds$ we first note that

$$q(s) = \Phi^\top(t, s)(I - \mathcal{N}(t, s; R, z)\mathcal{N}^{-1}(t, t-\sigma; R, z))h =$$

$$\Phi^\top(t, s)\mathcal{N}(s, t-\sigma; R, z)\mathcal{N}^{-1}(t, t-\sigma; R, z)h$$

[1]We used the obvious equality $\Phi(\tau, t-\sigma) = \Phi(\tau, t)\Phi(t, t-\sigma)$

for $s \geq t-\sigma$ and so

$$\int_{t-\sigma}^{t} q^\top Q q ds =$$

$$h^\top \mathcal{N}^{-1}(t,t-\sigma;R,z)W(t,t-\sigma)\mathcal{N}^{-1}(t,t-\sigma;R,z)h$$

with

$$W(t,t-\sigma) =$$

$$\int_{t-\sigma}^{t} \mathcal{N}(s,t-\sigma;R,z)\Phi(t,s)Q\Phi^\top(t,s)\mathcal{N}(s,t-\sigma;R,z)ds$$

Recalling that $\Phi(t,s)\Phi^\top(t,s) = I$ for skew-symmetric systems we get that:

$$W(t,t-\sigma) \leq \int_{t-\sigma}^{t} \lambda_{\max}(Q)\mathcal{N}^2(s,t-\sigma;R,z)ds.$$

Finally, we note that $\mathcal{N}^2(s,t-\sigma;R,z) \leq \mathcal{N}^2(t,t-\sigma;R,z)$ hence $W(t,t-\sigma) \leq \mathcal{N}^2(t,t-\sigma;R,z)\int_{t-\sigma}^{t}\lambda_{\max}(Q)ds$ and so $\int_{t-\sigma}^{t} z^\top Q z ds \leq \|h\|^2 \int_{t-\sigma}^{t} \lambda_{\max}(Q)ds$. This and (16) completes the proof. ■

The straightforward consequence of Corollary 3.3 is practical fixed-time convergence of the Riccati equation.

Corollary 3.4. *Let the conditions of Corollary 3.3 hold true. Then*

$$\Omega = \{P \in \mathbb{S}^n : 0 < P \leq (\bar{q}\sigma + (\underline{r}\alpha)^{-1})I\}$$

*is globally uniformly fixed-time attractive set of the system (10) with $T_{\max} = \sigma$.*

By using the same LQR-based argument as in the proof of Corollary 3.3 one can demonstrate the practical fixed-time convergence for $P^{-1}$:

Corollary 3.5. *Let $z$ and $P$ solve (9)-(10) on $[t_0, +\infty)$. Then, for any $\tilde{\sigma} > 0$ we have:*

$$\lambda_{\max}(P^{-1}(t)) \leq \bar{p}_{-1} := \frac{1}{\tilde{\sigma}\underline{q}} + \tilde{\sigma}\bar{c}\bar{r} \tag{17}$$

*Proof.* Given $z$ let us denote

$$\mathcal{N}(t,s,Q) = \int_{s}^{t} \Phi(t,\tau)Q(\tau)\Phi^\top(t,\tau)d\tau$$

and consider the following standard LQR control problem

$$\dot{q} = B(z)q + u, \quad q(t) = h,$$

$$J(u) = q(t_0)^\top P^{-1}(0)q(t_0) + \int_{t_0}^{t} u^\top Q^{-1}u + q^\top C^\top RCqd\tau.$$

Since $W(t) = P^{-1}(t)$ satisfies

$$\dot{W} = -B^\top(z)W - WB(z) - WQW + C^\top RC, W(t_0) = P^{-1}(0).$$

then $\tilde{u} = Q^{-1}Wq$ minimizes the functional $J$ and $J(\hat{u}) = h^\top W(t)h$. For any other control $u$ we have $J(\hat{u}) \leq J(u)$. Let us select $u$ as follows: set $u(s) = Q(s)\Phi^\top(s,t)\mathcal{N}^{-1}(t, t-\tilde{\sigma}, Q)h$ for $s \in [t-\tilde{\sigma}, t]$ and $u(s) = 0$ for $s \in [t_0, t-\tilde{\sigma}]$. Let us show that for this $u$ we have $q(s) = 0$ for all $s \in [t_0, t-\tilde{\sigma}]$. Since for $s \in [t-\tilde{\sigma}, t]$ we have

$$q(s) = \Phi(t,s)h - \int_s^t \Phi(s,\tau)u(\tau)d\tau =$$

$$\Phi(t,s)\left(I_n - \int_s^t \Phi(t,\tau)Q(\tau)\Phi^\top(\tau,t)d\tau\mathcal{N}^{-1}(t,t-\tilde{\sigma},Q)\right)h =$$

$$\Phi(t,s)\left(I_n - \mathcal{N}(t,s,Q)\mathcal{N}^{-1}(t,t-\tilde{\sigma},Q)\right)h =$$

$$\Phi(t,s)\left(\mathcal{N}(t,t-\tilde{\sigma},Q) - \mathcal{N}(t,s,Q)\right)\mathcal{N}^{-1}(t,t-\tilde{\sigma},Q)h =$$

$$\Phi(t,s)\mathcal{N}(s,t-\tilde{\sigma},Q)\mathcal{N}^{-1}(t,t-\tilde{\sigma},Q)h$$

then $q(s) = 0$ for $s \in [t_0, t-\tilde{\sigma}]$. Hence,

$$\int_{t_0}^t q^\top C^\top RCqds =$$

$$h^\top\mathcal{N}^{-T}(t,t-\tilde{\sigma},Q)V(t,t-\tilde{\sigma})\mathcal{N}^{-1}(t,t-\tilde{\sigma},Q)h$$

where

$$V(t,t-\tilde{\sigma}) =$$

$$\int_{t-\tilde{\sigma}}^t \mathcal{N}(s,t-\tilde{\sigma},Q)\Phi^\top(t,s)C^\top RC\Phi(t,s)\mathcal{N}(s,t-\tilde{\sigma},Q)ds.$$

Since $\Phi^\top(t,s)\Phi(t,s) = I$ for skew-symmetric matrices then

$$V(t,t-\tilde{\sigma}) \leq \bar{r}\bar{c}\int_{t-\tilde{\sigma}}^t \mathcal{N}^2(s,t-\tilde{\sigma},Q)ds.$$

Taking into account $N^2(s,t-\tilde{\sigma},Q) \leq N^2(t,t-\tilde{\sigma},Q)$ we derive

$$\int_{t_0}^t q^\top C^\top RCqds \leq \tilde{\sigma}\bar{c}\bar{r}\|h\|^2.$$

On the other hand, we have

$$\int_{t_0}^t u^\top Q^{-1}uds =$$

$$h^\top\mathcal{N}^{-1}(t,t-\tilde{\sigma},Q)\int_{t-\tilde{\sigma}}^t \Phi(s,t)Q(s)\Phi^\top(s,t)ds\mathcal{N}^{-1}(t,t-\tilde{\sigma},Q)h$$

and

$$\int_{t_0}^{t} u^\top Q^{-1} u ds = h^\top \mathcal{N}^{-1}(t, t-\tilde{\sigma}, Q) h \le \frac{\|h\|^2}{\tilde{\sigma} \underline{q}}.$$

Hence, we derive

$$J(\hat{u}) = h^\top W(t) h \le J(u) \le \frac{\|h\|^2}{\tilde{\sigma}\underline{q}} + \tilde{\sigma}\bar{c}\bar{r}\|h\|^2.$$

∎

*Remark* 3.6. Note that for the case of observable LTI systems the corresponding Riccati equation (10) has the property of the practical fixed-time convergence as defined above in 3.4.

**3.3. Observer design for bilinear equations.** In this section we demonstrate that the practical fixed-time convergence of the Riccati equation implies that (9) is an exponentially convergent observer for (7). To make this statement more precise we introduce

Lemma 3.7. *Let $x$ solve* (7)*, i.e.*

$$\dot{x} = B(x)x, \quad x(0) = x_0, \quad \|x_0\| = \theta > 0$$

*and let $e = x - z$ provided $z, P$ solve* (9) *and* (10)*. Then*

$$\dot{e} = (B(z) - PC^\top RC)e + B_1(x)e, e(0) = e_0, \tag{18}$$

$$B_1(x) = \left[\, B(\psi_1)x \; \dots \; B(\psi_n)x \,\right] \tag{19}$$

$$\|B_1(x)e\| \le \theta\beta\|e\|, \quad \beta^2 = \sum_{i=1}^{n} \|B(\psi_i)\|^2 \tag{20}$$

*Proof.* Upon differentiating $e$ we find:

$$\dot{e} = B(x)x - B(z)z - PC^\top R(y - Cz) =$$

$$B(z)e + B(e)x - PC^\top RCe.$$

By using the linearity of $B$ we get:

$$B(e)x = \sum_{i=1}^{n} (e^\top\psi_i)B(\psi_i)x = B_1(x)e.$$

To prove (20) we note that $\|x(t)\| = \|x(t_0)\| = \theta$ since $B^\top(x) = -B(x)$ and so $x \top \dot{x} = x^\top B(x)x = 0$. Then $\|B_1(x)e\| \le \|e\|\|x\| \left(\sum_{j=1}^{n} \|B(\psi_i)\|^2\right)^{\frac{1}{2}}$. ∎

In what follows we show that the estimation error $e$ decays to 0 exponentially fast for any initial condition $e(0)$ provided the Riccati equation is practically fixed-time stable.

Theorem 3.8. *Let $z, P$ solve* (9) *and* (10)*, and assume that $z(t_0)$ is chosen so that the gramian $\mathcal{N}(t, t-\sigma, z, R)$ along the trajectory $z$ of* (9) *verifies the following unifrom observability condition:*

$$\exists \alpha, \sigma > 0: \quad \alpha I < \mathcal{N}(t, t-\sigma; I, z), \quad \forall t > t_0 + \sigma.$$

*If the design parameters $\underline{q}, \bar{q}$, $\underline{r}, \bar{r}$ and $\tilde{\sigma}$ together with the given data $\bar{c}$ and $\theta, \beta$ verify the following inequality:*

$$\gamma := \frac{\underline{q}}{2\bar{p}} - \theta\beta(\bar{p}\bar{p}_{-1})^{\frac{1}{2}} > 0 \tag{21}$$

$$\bar{p}_{-1} = \frac{1}{\tilde{\sigma}\underline{q}} + \tilde{\sigma}\bar{r}\bar{c}, \quad \bar{p} = \frac{1}{\underline{r}\alpha} + \sigma\bar{q} \tag{22}$$

*then*

$$\|e(t)\| \le \|e(\tilde{t}_0)\|(\bar{p}\bar{p}_{-1})^{\frac{1}{2}} \exp\{-\gamma(t-\tilde{t}_0)\}, \tag{23}$$

*where* $\tilde{t}_0 = t_0 + \max\{\sigma, \tilde{\sigma}\}$.

*Proof.* Proof relies upon the following simple steps:

1) consider $B_1(x)e$ as an unknown perturbation in the error equation (18),
2) note that by (20) this perturbation is Lipschitz in $e$,
3) demonstrate that the solutions of the unperturbed equation

$$\dot{e}_1 = B(z)e_1 - PC^\top RCe_1, e_1(0) = e_0\,. \tag{24}$$

exponentially converge to 0 for $t \ge t_0 + \max\{\sigma, \tilde{\sigma}\}$

4) prove that the Lipschitz perturbation preserves the property 3)

Points 1) and 2) do not require any proof. Let us demonstrate 3). To this end, recall that $P$ is invertible by Theorem 3.1, and that

$$\dot{P}^{-1} = -P^{-1}B(z) - B^\top(z)P^{-1} - P^{-1}QP^{-1} + C^\top RC, \quad (*)$$

$P^{-1}(t_0) = P_0^{-1}$. Let us show that $P^{-1}$ is the Lyapunov function for the equation (24). It is not hard to see, by using $(*)$ and (24), that $\frac{d}{dt}e_1^\top P^{-1}e_1 = -e_1^\top(C^\top RC + P^{-1}QP^{-1})e_1$. Hence, for $t \ge \tilde{t}_0$ we have

$$\begin{aligned}
\frac{d}{dt}e_1^\top P^{-1}e_1 &\le -e_1^\top(P^{-1}QP^{-1})e_1 \\
&\le -\lambda_{\min}(Q)(P^{-\frac{1}{2}}e_1)^\top P^{-1}(P^{-\frac{1}{2}}e_1) \\
&\le -\underline{q}\lambda_{\min}(P^{-1})e_1^\top P^{-1}e_1 \\
&= -\frac{\underline{q}}{\lambda_{\max}(P)}e_1^\top P^{-1}e_1 \le -\frac{\underline{q}}{\bar{p}}e_1^\top P^{-1}e_1
\end{aligned}$$

provided $\bar{p} = \frac{1}{\underline{r}\alpha} + \sigma\bar{q}$ (recall that $\lambda_{\max}(P) \le \bar{p}$ by (15)). Hence, by Gronwall-Bellman lemma:

$$e_1^\top(t)P^{-1}(t)e_1(t) \le e_1^\top(s)P^{-1}(s)e_1(s)\exp\{-\frac{\underline{q}(t-s)}{\bar{p}}\}\,.$$

By (15), $e_1^\top P^{-1} e_1 \geq \frac{e_1^\top e_1}{\bar{p}}$ and, by (17), $e_1^\top P^{-1} e_1 \leq \bar{p}_{-1} e_1^\top e_1$, hence we get that:

$$\|e_1(t)\|^2 \leq \bar{p}\bar{p}_{-1} \|e_1(s)\|^2 \exp\{-\frac{q(t-s)}{\bar{p}}\}\,, \tag{25}$$

for any $s \in [\tilde{t}_0, t]$. To demonstrate point 3) let us denote by $\tilde{\Phi}$ the fundamental matrix of (24). Then $e_1(t) = \tilde{\Phi}(t,s) e_1(s)$, and by (25) it follows that

$$\|\tilde{\Phi}(t,s) e_1(s)\| \leq (\bar{p}\bar{p}_{-1})^{\frac{1}{2}} \|e_1(s)\| e^{-\frac{q}{2\bar{p}}(t-s)}$$

which in turn implies that

$$\|\tilde{\Phi}(t,s)\| \leq (\bar{p}\bar{p}_{-1})^{\frac{1}{2}} e^{-\frac{q}{2\bar{p}}(t-s)}. \tag{26}$$

for $t \geq s \geq \tilde{t}_0$. Hence, the estimation error given by the equation (24) converges exponentially to zero for $t > t_0 + \max\{\sigma, \tilde{\sigma}\}$.

Now, to prove point 4) we use classical results on shift of Lyapunov spectrum of uniformly exponentially stable LTV systems by Lipschitz perturbations (see Section 2). Specifically, let $\gamma$ be defined as in theorem's statement. Recall from Lemma 3.7 that:

- the estimation error equation (18) can be otained by adding a linear perturbation $B_1(x)e$ to the LTV (24) (point 1 above),
- the matrix of the perturbation is bounded: $\|B_1(x)\| \leq \theta\beta$ (point 2) above),
- the fundamental matrix of the unperturbed equation (24) verifies (26) (point 3) above).

Hence, by Theorem 4.4.6 from Section 2 the fundamental matrix $\Phi(t,s)$ of the perturbed equation (18) admits the following estimate: $\|\Phi(t,s)\| \leq (\bar{p}\bar{p}_{-1})^{\frac{1}{2}} e^{-\gamma(t-s)}$, and so (23) holds. This completes the proof. ∎

**4. Numerical example.** Consider the ODE obtained by discretizing the Burgers(-Hopf) equation $u_t = -\frac{\partial_\xi u^2}{2}$ on $(0,1)$ with periodic boundary conditions by using the finite difference scheme:

$$\dot{u}_i = -\frac{n}{6}\left(u_i(u_{i+1} - u_{i-1}) + (u_{i+1}^2 - u_{i-1}^2)\right) \tag{27}$$

taken on a periodic lattice ($i = 1 \ldots n$, $u_{-1} = u_n$, $u_{n+1} = u_1$) which has the properties that

- the quadratic energy $\sum_i u_i^2$ is conserved, implying that every sphere in $\mathbb{R}^n$ is invariant under the motion of the system and $\|u\|$ is constant,
- the trace of the Jacobian of the r.h.s. of (27) is zero, implying that the flow conserves the volume of the phase element.

Denoting $x = (u_1 \dots u_n)^\top$ and setting[2] $D = \{D_{ij}\}_{i,j=1}^n$ with $D_{ji} = -D_{ij}$ for $j \le i$, and $D_{ii+1} = 1$, $D_{ij} = 0$ for $j > i+1$ except for $D_{1n} = -1$, we can rewrite (27) as follows: $\dot{x} = B(x)x$ with $B(x) = -\frac{n}{6}(\operatorname{diag}(x)D + D\operatorname{diag}(x))$. Clearly, $B^\top(x) = -\frac{n}{6}(D^\top \operatorname{diag}(x) + \operatorname{diag}(x)D^\top) = -B(x)$ since $D^\top = -D$.

We set $n = 8$ and perform three experiments for $m = 5$, $m = 4$ and $m = 3$:

$m = 5$ we measure $u_1$, $u_2$, $u_4$ and $u_6$ and $u_8$ – $C_5$ is a $5 \times 8$ matrix represented by 1st, 2nd, 4th, 6th and 8th rows of $I_8$,

$m = 4$ we measure $u_2$, $u_4$ and $u_6$ and $u_8$, and define $C_4$ accordingly,

$m = 3$ we measure $u_2$, $u_4$ and $u_6$, and define $C_3$ accordingly.

To perform the above experiments, we first generate 10 different "true" initial conditions by taking $x(0) \sim U(0, 1)$, remove the mean (to account for the periodicity), and initialize the filter by perturbing the "true" $x(0)$: $z_i(0) = x_i(0) + q$, $q$ is drawn from the standard normal distribution. Then, for each matrix $C \in \{C_5, C_4, C_3\}$ and for each $x(0)$ and $z(0)$ we solve the state equation (7), the filter (9), and Ricatti equation (10) simultaneously by RK4 with $\Delta t = 5 \times 10^{-4}$. In all the simulations we use $P(0) = I_8$, $Q = \bar{q}I_8$ with $\bar{q} = 5 \times 10^4 + 1$, $R = \bar{r}I_m$ with $\bar{r} = 1 \times 10^2$.

Finally, each simulation was performed until time instant $T$, specific to the simulation, when the numerical convergence has been observed, i.e. $\|x(T) - z(T)\| < 1 \times 10^{-16}$. The results are given in Figs. 1, 2 and 3 respectively.

As it can be seen from Fig. 1 and 2 the convergence takes a similar pattern: transient phase on an interval $t_k, t_k + \sigma_k$ (during this interval the error either levels off or even increases) is followed by an exponential decay of the estimation error, which in turn is followed by another transient phase until numerical convergence is observed. In the case of $m = 6$ the transient phases are less pronounced in that the convergence is almost uniform (see Fig. 1). For $m = 4$ the non-uniformity becomes more pronounced yet, the decay is largely uniform (see Fig. 2). Finally, for the case of $m = 3$ the exponential convergence is very non-uniform (see Fig. 3). We stress that the relative errors in initial conditions, given in the legend of each figure, are rather high (up to 400 %).

In our opinion, the non-uniform convergence (i.e. the "switching" between transient phases and convergence) could be explained as follows: as noted in Theorem 3.8, the uniform exponential convergence begins after time[3] $\sigma$, and the error decays exponentially at the same rate $\gamma > 0$ after that time provided $\sigma$ and $\alpha$ do not change; however, the actual values of $\alpha$ and $\sigma$ are unknown, and the conditions of the Theorem may be violated "from time to time" in that the discrete numerical system looses observability during transient phases, but it regains it again leading to the observed non-uniform convergence pattern. This effect is less pronounced for $m = 6$ and $m = 4$ but is very clear for $n = 3$.

[2] MATLAB instruction for creating $D$: $\mathrm{D = diag(ones(n-1,1),1) - diag(ones(n-1,1),-1); D(1,n) = -1; D(n,1) = 1;}$

[3] This is the time required for $\mathcal{N}$ to become invertible.

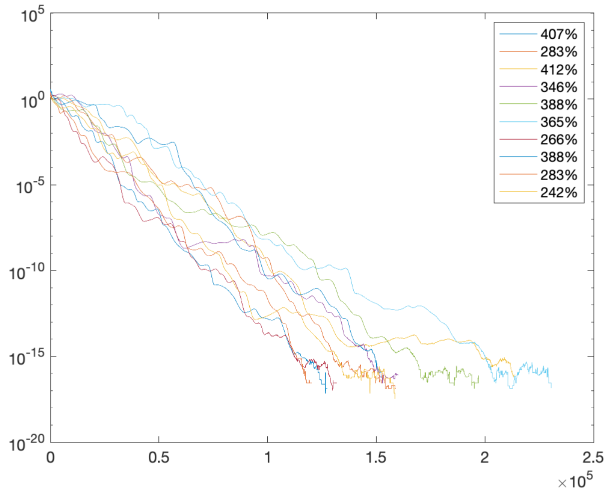


Figure 1: Convergence of filter for the discretized Burgers equation (27) with $C_6$ ($m = 6$), showing the estimation errors for a 10-member ensemble of perturbed initial conditions (log-scale). Legend displays relative errors in initial conditions.

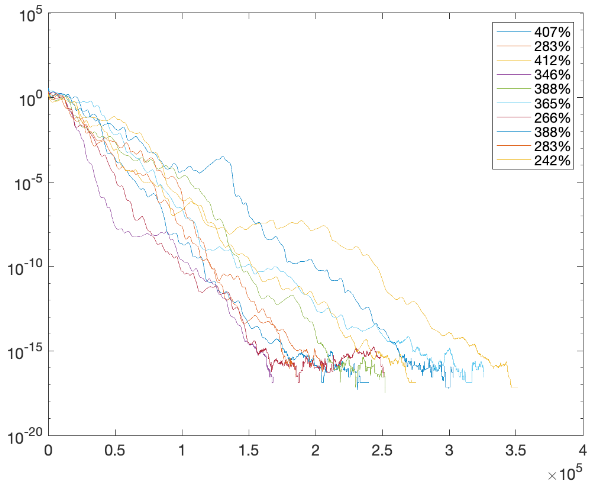


Figure 2: Convergence of filter for the discretized Burgers equation (27) with $C_4$ ($m = 4$, log-scale).

**5. Conclusion.** The paper reveals an important relation between uniform complete observability, fixed time convergence for differential Riccati equations and observer design. This relation, on the other hand, requires further exploration as to how to actually compute parameters of the observer in practise, and how to deal with the case of non-trivial model error.

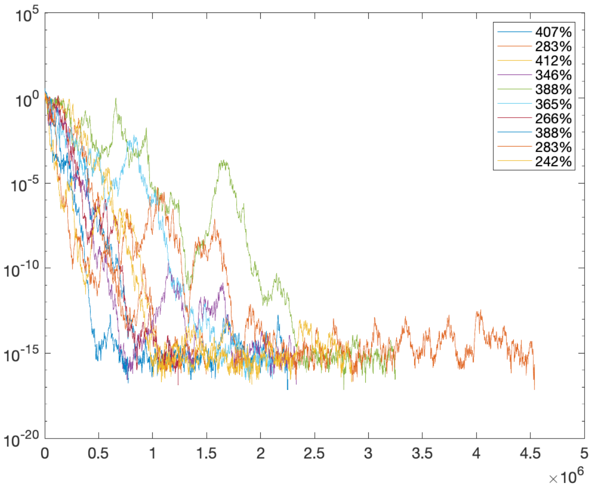


Figure 3: Convergence of filter for the discretized Burgers equation (27) with $C_3$ ($m = 3$, log-scale).

## REFERENCES


[Bucy(1972)] Bucy, R.S. (1972). The riccati equation and its bounds. *Journal of computer and system sciences*, 6(4), 343–353.

[Fliess et al.(1995)Fliess, Lévine, Martin, and Rouchon] Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1995). Flatness and defect of non-linear systems: introductory theory and examples. *International journal of control*, 61(6), 1327–1361.

[Frank and Zhuk(2018)] Frank, J. and Zhuk, S. (2018). A detectability criterion and data assimilation for nonlinear differential equations. *Nonlinearity*, 31(11), 5235.

[Hermann and Krener(1977)] Hermann, R. and Krener, A. (1977). Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22(5), 728–740. doi:10.1109/TAC.1977.1101601.

[Isidori(1995)] Isidori, A. (1995). *Nonlinear Control Systems*. Springer-Verlag, N. Y. Inc.

[Krener(1980)] Krener, A.J. (1980). Kalman-Bucy and minimax filtering. *IEEE Trans. on Autom. Control*, 25(2), 291–292. doi:10.1109/TAC.1980.1102301.

[Polyakov(2012)] Polyakov, A. (2012). Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Trans. Autom. Cont.*, 57(8), 2106–2110.

[Tranninger et al.(2019)Tranninger, Seeber, Zhuk, Steinberger, and Horn] Tranninger, M., Seeber, R., Zhuk, S., Steinberger, M., and Horn, M. (2019). Detectability analysis and observer design for linear time varying systems. *IEEE Control Systems Letters*, 4(2), 331–336.

[1] L. Y. Adrianova. *Introduction to linear systems of differential equations*, volume 146 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1995.

[2] J. S. Baras, A. Bensoussan, and M. R. James. Dynamic observers as asymptotic limits of recursive filters: Special cases. *SIAM Journal on Applied Mathematics*, 48(5):1147–1158, 1988.

[3] M. Bardi and I. Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi Equations*. Birkhäuser, 1997.

[4] G. Evensen. The ensemble kalman filter: Theoretical formulation and practical implementation. *Ocean Dynamics*, (53):343–367, 2003.

[5] T. Tchrakian, J. Frank, and S. Zhuk. Exponentially convergent data assimilation algorithm for navier-stokes equations. In *Proc. of American Control Conference*, 2017.

[6] M. Ghil, S. Cohn, J. Tavantzis, K. Bube, and E. Isaacson. Applications of estimation theory to numerical

weather prediction. In *Dynamic meteorology: Data assimilation methods*, pages 139–224. Springer, 1981.

[7] I. Gihman and A. Skorokhod. *Introduction to the Theory of Random Processes*. Dover Books on Mathematics. Dover, 1997.

[8] J. S. Baras and A. Kurzhanski. Nonlinear filtering: The set-membership and the $H_\infty$ techniques. In *Proc. 3rd IFAC Symp.Nonlinear Control Sys.Design*. Pergamon, 1995.

[9] T. Tchrakian and S. Zhuk. A macroscopic traffic data assimilation framework based on Fourier-Galerkin method and minimax estimation. *IEEE Tran. Intel. Transp. Sys.*, (99):1–13, 2014. special issue.

[10] S. Zhuk and T. Tchrakian. Parameter estimation for euler equations with uncertain inputs. In *Proc. of IEEE Conference on Decision and Control*, 2015.

[11] S. Boccalettia, J. Kurths, G. Osipov, D. Valladares, and C. Zhou. The synchronization of chaotic systems. *Physics Reports*, 366:1–101, 2002.

[12] M. Bocquet, K. S. Gurumoorthy, A. Apte, A. Carrassi, C. Grudzien, and C. K. Jones. Degenerate kalman filter error covariances and their convergence onto the unstable subspace. *SIAM/ASA Journal on Uncertainty Quantification*, 5(1):304–333, 2017.